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Localization of the relativistic centre-of-mass from internal dynamics

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Abstract

The localization of the centre-of-mass (CM) of a two-body electromagnetic/gravitational system in the post-Newtonian approximation is shown to depend, in general, on the internal configuration of the system via the corresponding Newtonian Runge–Lenz vector. The requirement that the CM should not depend on the internal configuration then uniquely determines the CM coordinate in terms of the Runge–Lenz vector. A similar result is found for fully relativistic non-interacting particles. We conclude that consideration of the CM of relativistic systems should involve the Runge–Lenz symmetry explicitly, as an essential part of the internal symmetries of the system.

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1. Introduction

Seeking for a proper definition of the centre-of-mass (CM) of extended or composite relativistic systems is an essential part of the effort to separate the internal dynamics of such systems from their global motion. From the beginning [1] it was clear that various and different definitions are possible, and that the relativistic CM may be defined either as a 4-vector in Minkowsky spacetime or as a 3-vector on spacelike hyperplanes; that even as a 4-vector it may be either manifestly covariant (observer-independent) or not; and that it may be either a canonical coordinate (with vanishing commutation relations or Poisson brackets (PB) between all its components) or not (see [2] for a recent publication with an extensive bibliography covering the subject's history).

In the following we concentrate on the manifestly covariant CM coordinate for non-quantum systems. If the system is Lorentz–Poincaré symmetric with conserved total linear momentum P^μ, then the CM motion is simply a straight line parallel to P^μ. However, the spatial position of the CM relative to the CM system is ambiguously defined, not because it may be subject to arbitrary translations (which is always the case, but this is only a geometrical symmetry) but because definitions are possible which are *dynamically* different [1–4].

The differences between these definitions are termed dynamical, because they involve dependence on the internal dynamics of the system. Then, in general, a change in the internal configuration of a system will inevitably induce a change in the resultant CM.

So far, the different options for the relativistic CM have been formulated only in terms of the internal angular momentum of the system. However, the analysis of internal dynamics of rotationally symmetric systems involves not only rotational symmetry but rather the larger symmetry generated together by both the internal angular momentum and the so-called Runge–Lenz[1] vector [6–10], and this includes also various relativistic systems and models, both non-quantum [7, 11–14] and quantum [15–17]. But it seems that the possible relation between the relativistic CM and the Runge–Lenz symmetry was ignored over the years.

The purpose of this paper is to demonstrate, first, that the computation of the relativistic CM coordinate explicitly involves, at least for two-body systems, the Runge–Lenz vector. Explicit expressions for relativistic Runge–Lenz vectors were found for single particles in centrally symmetric potentials (fixed centre) [7, 11], and for couples of particles in the post-Newtonian approximation [12] or in models with non-realistic but simple-to-handle interactions [13, 14]. Corresponding expressions for fully relativistic systems with real interactions are not yet known, but the problem is fully solvable in the post-Newtonian approximation, in which we compute the CM coordinate for electromagnetic or gravitational systems. There the CM is indeed found to be dependent, in general, on the internal configuration of the system via the Newtonian Runge–Lenz vector. Then, introducing the requirement that the CM coordinate be independent of the internal configuration uniquely fixes its form. This result is then corroborated for a two-body fully relativistic system of non-interacting particles. Having fixed the form of the CM coordinate it is shown that it cannot be canonical. Implications and possible consequences of these results for fully relativistic systems are then discussed.

2. Internal symmetries and the general form of the spatial CM coordinate

For closed relativistic systems of particles, with conserved total linear momentum P^μ and total angular momentum $J^{\mu\nu}$, the trajectory in Minkowski spacetime[2] of their CM may always be written as

$$X^\mu(\tau) = R^\mu + \tau \cdot \frac{P^\mu}{M} \tag{1}$$

where $M = \sqrt{-P^2}/c$ is the total mass of the system, R^μ is a constant 4-vector, the spatial CM coordinate relative to the origin of the CM-frame and τ is the CM proper time, a Lorentz scalar. Appropriately fixing the zero of τ, R^μ may be assumed orthogonal to P^μ without loss of generality, $R \cdot P = 0$.

A full and complete definition of the CM coordinate (1) requires separate definition of R^μ and τ. The definition of the proper-time τ as an observable was recently discussed in [18, 19]. Here we discuss the spatial part R^μ.

As already mentioned, the determination of R^μ turned out to be ambiguous and different approaches to the relativistic CM issue have yielded a variety of expressions for R^μ. The

[1] This vector should actually be named after the predecessors of Runge and Lenz [5], but for the sake of simplicity the common name will be used in the following.

[2] In the following we consider dynamics described in a Minkowski spacetime with metric tensor $g_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$, $\mu, \nu = 0, 1, 2, 3$.

minimal relativistic Lorentz-covariant generalization of the Newtonian expression, also known as the centre-of-inertia [2–4], is

$$R^{\mu} = -\frac{J^{\mu\nu}P_{\nu}}{M^2c^2}. \tag{2}$$

For the general expression it is convenient to introduce a vector Q^{μ} which incorporates all the differences between the various definitions, so that R^{μ} becomes

$$R^{\mu} = -\frac{J^{\mu\nu}P_{\nu}}{M^2c^2} + Q^{\mu}, \tag{3}$$

and discuss its properties. Clearly $Q \cdot P = 0$, so that Q^{μ} is a spacelike vector, fully defined in the spatial part of the CM reference frame. Since the translational properties of R^{μ} are all contained in the first term $-J^{\mu\nu}P_{\nu}/M^2c^2$ it follows that Q^{μ} is invariant under global uniform translations and may only depend on the relative positions of the particles. Thus we conclude that the differences between the various CM definitions are incorporated in different dependences on the internal dynamics of the system.

In particular, in a two-body system there are six internal degrees of freedom. Four of them are contained in the total energy and the internal angular momentum vector $\vec{\ell} = \vec{r} \times \vec{p}$ ($\vec{r}$ and $\vec{p}$ defined as in equation (4) below). The other two are contained in the Runge–Lenz vector $\vec{K}$ or its generalization which exists for any rotationally symmetric two-body system [6, 7], and whose existence and constancy is a manifestation of an internal dynamical symmetry (classically, the Runge–Lenz vector is the generator of the canonical transformations that take the system from one orbit to another, with the same energy). $\vec{Q}$, as a constant 3-vector depending only on the relative coordinates of the particles, must then be determined only by $\vec{K}$ and $\vec{\ell}$.

3. The vector $\vec{Q}$ in the post-Newtonian approximation

We now proceed to show, in the post-Newtonian approximation of an electrical or gravitational two-body system, that the general definition equation (3) of R^{μ} depends, independently of Q^{μ}, on the Newtonian Runge–Lenz vector. This, in turn, implies that R^{μ} depends, in general, on the internal configuration of the system; and if we expect the CM to be independent of the internal configuration then this poses a requirement that uniquely determines Q^{μ} in terms of the Runge–Lenz vector.

The demonstration is simple, based on a procedure first introduced by Dahl [20]. Consider a two-particle system, with masses m_1, m_2, possible electrical charges e_1, e_2, spatial coordinates $\vec{x}_1, \vec{x}_2$ and linear momenta $\vec{p}_1, \vec{p}_2$. We also introduce notations for the total Newtonian mass $M_o = m_1 + m_2$ and the Newtonian reduced mass $\mu_o = m_1m_2/M_o$. In the CM system (defined by $\vec{P} = \vec{p}_1 + \vec{p}_2 = 0$, without fixing the origin) we define the internal canonical variables

$$\vec{r} = \vec{x}_1 - \vec{x}_2, \qquad \vec{p} = \vec{p}_1 = -\vec{p}_2. \tag{4}$$

Dahl's procedure consists in computing the post-Newtonian Lorentz boost in the CM frame, starting with the vector

$$\vec{R}_o = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{M_o}, \tag{5}$$

which formally looks like the Newtonian CM, but in the post-Newtonian approximation is not a constant of the motion. To avoid a possible (false) impression that the procedure and its consequent results depend on this particular choice of $\vec{R}_o$, we choose to start with a more general expression: let η_1 and η_2 be arbitrary constants, independent of $\vec{p}$, such that $\eta_1+\eta_2 = 1$, and let $\vec{R}_o$ be defined as

$$\vec{R}_o = \eta_1\vec{x}_1 + \eta_2\vec{x}_2. \tag{6}$$

These definitions allow us to express the particles' coordinates as

$$\vec{x}_1 = \vec{R}_o + \eta_2\vec{r}, \qquad \vec{x}_2 = \vec{R}_o - \eta_1\vec{r}. \tag{7}$$

Using the post-Newtonian velocities [21]

$$\begin{aligned}\vec{v}_1 &= \left(1 - \frac{p^2}{2m_1^2c^2}\right)\frac{\vec{p}}{m_1} + \frac{\kappa}{2rm_1m_2c^2}\left[(1+m_2\alpha)\vec{p} + \frac{(\vec{p}\cdot\vec{r})}{r^2}\vec{r}\right],\\ \vec{v}_2 &= -\left(1 - \frac{p^2}{2m_2^2c^2}\right)\frac{\vec{p}}{m_2} - \frac{\kappa}{2rm_1m_2c^2}\left[(1+m_1\alpha)\vec{p} + \frac{(\vec{p}\cdot\vec{r})}{r^2}\vec{r}\right],\end{aligned} \tag{8}$$

with $\kappa = e_1e_2$ and $\alpha = 0$ or $\kappa = -Gm_1m_2$ and $\alpha = 6/\mu_o$ for the electrical or gravitational case, respectively, the time derivative of $\vec{R}_o$ is found to be, after some algebra,

$$\begin{aligned}\frac{\mathrm{d}\vec{R}_o}{\mathrm{d}t} &= \eta_1\vec{v}_1 + \eta_2\vec{v}_2\\ &= \mu_o\left(\frac{\eta_1}{m_1} - \frac{\eta_2}{m_2}\right)\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} + \frac{m_1-m_2}{2M_om_1m_2c^2}\left[\frac{p^2\vec{p}}{\mu_o} + \frac{\kappa}{r}\vec{p} + \frac{\kappa(\vec{p}\cdot\vec{r})}{r^3}\vec{r}\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_o\left(\frac{\eta_1}{m_1} - \frac{\eta_2}{m_2}\right)\vec{r} + \frac{m_1-m_2}{2\mu_oM_o^2c^2}(\vec{p}\cdot\vec{r})\vec{p}\right].\end{aligned} \tag{9}$$

The virtue of this decomposition is that the coefficient of $\mathrm{d}\vec{r}/\mathrm{d}t$ is independent of $\vec{p}$, while the other term is independent of the η_as. Passing from second to third row in equation (9) employed the Newtonian equations of motion, which were sufficient because these expressions are already of order $1/c^2$. Integration of equation (9) then yields

$$\vec{R}_o = \vec{X}_o + \mu_o\left(\frac{\eta_1}{m_1} - \frac{\eta_2}{m_2}\right)\vec{r} + \frac{m_1-m_2}{2\mu_oM_o^2c^2}(\vec{p}\cdot\vec{r})\vec{p}, \tag{10}$$

with $\vec{X}_o$ being an arbitrary integration constant.

The post-Newtonian Lorentz boost in the CM frame [21] now becomes, using equation (7) ($N^i = J^{i0}$)

$$\begin{aligned}\vec{N} &= \sum_a\left(m_a + \frac{p^2}{2m_ac^2} + \frac{\kappa}{2rc^2}\right)\vec{x}_a\\ &= M\vec{R}_o + \left[\eta_2m_1 - \eta_1m_2 + \left(\frac{\eta_2}{m_1} - \frac{\eta_1}{m_2}\right)\frac{p^2}{2c^2} + \frac{(\eta_2-\eta_1)\kappa}{2rc^2}\right]\vec{r},\end{aligned} \tag{11}$$

where

$$M = M_o + \frac{p^2}{2\mu_oc^2} + \frac{\kappa}{rc^2} \tag{12}$$

is the post-Newtonian total mass in the CM coordinates. Then, substituting $\vec{R}_o$ from equation (10), it turns out that all the η_a-dependent terms cancel, resulting in

$$\vec{N} = M\vec{X}_o + \frac{m_2-m_1}{2\mu_oM_oc^2}\left[\left(p^2 + \frac{\mu_o\kappa}{r}\right)\vec{r} - (\vec{p}\cdot\vec{r})\vec{p}\right]. \tag{13}$$

The vector in the square brackets is easily recognized as the Runge–Lenz vector of the corresponding Newtonian system,

$$\vec{K} = \left(p^2 + \frac{\mu_o\kappa}{r}\right)\vec{r} - (\vec{p}\cdot\vec{r})\vec{p} = \vec{p}\times\vec{\ell} + \frac{\mu_o\kappa}{r}\vec{r} \tag{14}$$

and we finally obtain

$$\vec{N} = M\vec{X}_o + \frac{m_2 - m_1}{2\mu_o M_o c^2}\vec{K}. \tag{15}$$

This is Dahl's result, and it is evident that it does not depend on the particular choice of the η_a coefficients in equation (6).

The computation of the CM is now straightforward, since the vector R^μ in equation (3) has, in the CM system, only spatial components,

$$\vec{R} = \frac{\vec{N}}{M} + \vec{Q} \tag{16}$$

so that applying equation (15) it becomes

$$\vec{R} = \vec{X}_o + \frac{m_2 - m_1}{2\mu_o M_o^2 c^2}\vec{K} + \vec{Q}. \tag{17}$$

$\vec{X}_o$, the integration constant, serves to fix the origin of the CM reference frame. In the Newtonian limit we simply obtain $\vec{R} = \vec{X}_o$ and then may make the common choice $\vec{X}_o = 0$. Here (in the post-Newtonian case) we may again take $\vec{X}_o = 0$, because it is an integration constant, not a dynamical quantity, but in order to make also $\vec{R} = 0$ the identification

$$\vec{Q} = \frac{m_1 - m_2}{2\mu_o M_o^2 c^2}\vec{K} \tag{18}$$

is required.

It should again be emphasized that $\vec{X}_o$ is the constant value of $\vec{R}$, while as an observable, a function of the dynamical variables of the system, $\vec{R}$ is given by equation (16): in Newtonian dynamics any value may be attached to $\vec{R}$ without regard to the actual configuration in which the system is. In particular, even if the total energy and angular momentum are not known, the same value $\vec{R} = \vec{X}_o$ may be assumed for all possible configurations. However, the value of the Runge–Lenz vector, even if constant, depends on the particular configuration (e.g., orbits with different eccentricity). Then, if equation (18) is not satisfied, it implies that either the final value of $\vec{R}$ depends on the configuration, or that for any different configuration of the system the value of $\vec{X}_o$ must be adapted to obtain a desired value for $\vec{R}$. Therefore, equation (18) is necessary to make the determination of $\vec{R}$ configuration independent.

The vector $\vec{Q}$ is therefore found to be uniquely determined by the Runge–Lenz vector of the system. It is important to note that the Runge–Lenz vector was not imposed or introduced arbitrarily in any way into the preceding expressions, but it emerged naturally in expression equation (15) for the boost. It is interesting to note that although $\vec{Q}$ could have been any linear combination of $\vec{K}$, $\vec{K} \times \vec{\ell}$ and $\vec{\ell}$, it is simply proportional to $\vec{K}$ alone. We also note that Duviryak [14] reported a similar shift of the foci of the orbit relative to the assumed CM, proportional to the Runge–Lenz vector. He, however, considered the CM as being fixed by equation (2) and did not examine the implication of the shift on the definition of the CM.

4. Determination of $\vec{Q}$ for a relativistic pair of non-interacting particles

The post-Newtonian approximation is the simplest context in which we can show some relativistic behaviour. Since we do not know yet the Runge–Lenz-like vector for fully relativistic systems with real interactions, let us show how the foregoing procedure works for two free relativistic particles. The following discussion will also provide an insight regarding the independence of Dahl's procedure on the η_a coefficients.

In the CM system, with momenta and relative coordinates as in equation (4), energies $E_a = \sqrt{m_a^2 + p^2}$ and $\vec{R}_o$ defined as in equation (6) (in this section the convention $c = 1$ is assumed), the Lorentz boost is

$$\vec{N} = \sum_a E_a \vec{x}_a = E_1(\vec{R}_o + \eta_2 \vec{r}) + E_2(\vec{R}_o - \eta_1 \vec{r})$$
$$= M\left[\vec{R}_o + \left(\eta_2 \frac{E_1}{M} - \eta_1 \frac{E_2}{M}\right)\vec{r}\right], \tag{19}$$

where $M = E_1 + E_2$ is the total relativistic mass. Since

$$\frac{E_1}{M} + \frac{E_2}{M} = 1$$

let $f(p^2)$ be defined so that

$$\frac{E_1}{M} = \eta_1 + f(p^2), \qquad \frac{E_2}{M} = \eta_2 - f(p^2)$$

yielding

$$f(p^2) = \frac{E_1 - E_2}{2M} - \frac{\eta_1 - \eta_2}{2} = \frac{E_1^2 - E_2^2}{2M^2} - \frac{\eta_1 - \eta_2}{2}$$
$$= \frac{m_1^2 - m_2^2}{2M^2} - \frac{\eta_1 - \eta_2}{2}. \tag{20}$$

In the absence of relative motion, $\vec{p} = 0$, $f(p^2)$ becomes

$$f(0) = \frac{m_1^2 - m_2^2}{2M_o^2} - \frac{\eta_1 - \eta_2}{2} = \mu_o\left(\frac{\eta_2}{m_2} - \frac{\eta_1}{m_1}\right).$$

The dependence of $f(p^2)$ on the η_a coefficients is therefore only in $f(0)$, so that it may be decomposed into

$$f(p^2) = \frac{m_1^2 - m_2^2}{2}\left(\frac{1}{M^2} - \frac{1}{M_o^2}\right) + f(0) = \frac{(m_2 - m_1)\left(M^2 - M_o^2\right)}{2M_o M^2} + f(0).$$

Expressing p^2 in terms of the masses,

$$p^2 = \frac{\left(M^2 - M_o^2\right)[M^2 - (m_1 - m_2)^2]}{4M^2}$$

$f(p^2)$ becomes

$$f(p^2) = \frac{2(m_2 - m_1)p^2}{M_o[M^2 - (m_1 - m_2)^2]} + f(0) \tag{21}$$

and the Lorentz boost is finally obtained:

$$\vec{N} = M[\vec{R}_o + f(p^2)\vec{r}] = M\vec{R}_o + M\left\{\frac{2(m_2 - m_1)p^2}{M_o[M^2 - (m_1 - m_2)^2]} + f(0)\right\}\vec{r}. \tag{22}$$

The particles' velocities are

$$\vec{v}_1 = \frac{\vec{p}}{E_1}, \qquad \vec{v}_2 = -\frac{\vec{p}}{E_2},$$

with the relative velocity

$$\frac{d\vec{r}}{dt} = \vec{v}_1 - \vec{v}_2 = \frac{\vec{p}}{E_1} + \frac{\vec{p}}{E_2} = \frac{\vec{p}}{\mu},$$

where $\mu = E_1E_2/M$ is the relativistic reduced mass. The time derivative of R_o may be computed and written as

$$\frac{\mathrm{d}\vec{R}_o}{\mathrm{d}t} = \eta_1\vec{v}_1 + \eta_2\vec{v}_2 = \left(\eta_1\frac{E_2}{M} - \eta_2\frac{E_1}{M}\right)\frac{\vec{p}}{\mu} = -f(p^2)\frac{\vec{p}}{\mu}$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(0)\vec{r} + \frac{2(m_2-m_1)}{M_o[M^2-(m_1-m_2)^2]}(\vec{r}\cdot\vec{p})\vec{p}\right\} \tag{23}$$

taking into account the constancy of $\vec{p}$ and M. Integrating equation (23) we obtain

$$\vec{R}_o = \vec{X}_o - f(0)\vec{r} - \frac{2(m_2-m_1)}{M_o[M^2-(m_1-m_2)^2]}(\vec{r}\cdot\vec{p})\vec{p} \tag{24}$$

so that combining equation (19) and equation (24) into equation (16) yields

$$\vec{R} = \vec{X}_o + \frac{2(m_2-m_1)}{M_o[M^2-(m_1-m_2)^2]}[p^2\vec{r} - (\vec{r}\cdot\vec{p})\vec{p}] + \vec{Q}$$
$$= \vec{X}_o + \frac{2(m_2-m_1)}{M_o[M^2-(m_1-m_2)^2]}\vec{p}\times\vec{\ell} + \vec{Q}. \tag{25}$$

It is again evident that the arbitrariness of η_a, included only in $f(0)$, disappears from the final computation of $\vec{N}$ or $\vec{R}$, thus verifying the validity of these results. Dahl's choice for $\vec{R}_o$ in equation (5) with $\eta_a = m_a/M_o$ is just the simplest or most convenient one, for which $f(0) = 0$ and the $\vec{r}$-term ia absent from $\vec{R}_o$.

As above, $\vec{Q}$ is determined by the requirement that the sum of the last two terms in equation (25) vanishes, so we finally obtain

$$\vec{Q} = \frac{2(m_1-m_2)}{M_o[M^2-(m_1-m_2)^2]}\vec{p}\times\vec{\ell}. \tag{26}$$

Comparing with equation (14), $\vec{p}\times\vec{\ell}$ is the Runge–Lenz vector of the non-interacting couple, and it is the correct expression in the absence of interactions also in the full relativistic case. In the non-relativistic limit equation (26) reduces to equation (18), without the interaction term in the Runge–Lenz vector. Since the coefficient in equation (26) depends only on the particles' masses and the total mass, it is expected that the relation

$$\vec{Q} = \frac{2(m_1-m_2)}{M_o[M^2-(m_1-m_2)^2]}\vec{K} \tag{27}$$

will be valid for all relativistic systems, including interactions.

5. The non-canonicity of the CM coordinate

A long-standing issue of the relativistic CM coordinate X^μ (position operator in the quantum case) is that in general $[X^\mu, X^\nu] \neq 0$, where the brackets imply Poisson brackets (PB) or commutation relations for non-quantum or quantum systems, respectively, unlike what is expected from a canonical coordinate [1–3, 22]. The only instance in which these self-PB or commutators vanish is when there is a fully relativistic spin tensor $S^{\mu\nu}$ with $O(3,1)$ symmetry [3].

This self-non-commutativity of the CM coordinate was interpreted as X^μ not being a canonical coordinate, as opposed to the non-relativistic case. The definition of Q^μ affects, of course, the self-commutation of X^μ. In particular it may be shown that a necessary condition for $[X^\mu, X^\nu] = 0$ is that

$$[Q^\mu, Q^\nu] = -\frac{\ell^{\mu\nu}}{M^2c^2}, \tag{28}$$

where

$$\ell^{\mu\nu} = -(J^{\mu\nu}P^{\lambda} + J^{\lambda\mu}P^{\nu} + J^{\nu\lambda}P^{\mu})\frac{P_{\lambda}}{M^2c^2} \tag{29}$$

is the spatial internal angular momentum tensor of the system, dually related to the vector $\vec{\ell} = \vec{r} \times \vec{p}$ in the CM system. However, with Q^{μ} given by equation (18) we obtain

$$[Q^i, Q^j] = -\frac{(m_1 - m_2)^2(M - M_o)}{2\mu_o M_o^4 c^2}\ell^{ij} \neq -\frac{\ell^{ij}}{M^2c^2} \tag{30}$$

$(i, j = 1, 2, 3)$ so that indeed $[X^{\mu}, X^{\nu}] \neq 0$ except for a pair with equal masses. In the quantum case this result implies that the spatial components of X^{μ} in the plane defined by $\ell^{\mu\nu}$ (classical plane of motion) cannot be fixed simultaneously.

6. Discussion

The present paper brings together and combines two subjects which so far were regarded as completely distinct: the determination of the CM of composite relativistic systems, on the one hand, and the Runge–Lenz vector on the other hand. From the two cases which were presented it follows that, unlike all the previous approaches to the relativistic CM issue, the Runge–Lenz vector naturally emerges and thus plays a major role in the determination of the relativistic CM. The common view has so far held that the CM coordinate of composite relativistic systems should be constructed of P^{μ} and $J^{\mu\nu}$ alone [2]; our results, together with those in [18, 19], challenge this view.

The Runge–Lenz vector is a constant of the motion in general centrally symmetric systems, depending only on the internal dynamics of the system [5–8]. It generates, together with the internal angular momentum, $SO(4)$ or $SO(3, 1)$ symmetry groups which contain the internal rotational symmetry as a subgroup. In classical (non-quantum) systems, knowledge of the Runge–Lenz vector amounts to having a full solution for the configuration of the system (details of orbit, etc); in simple quantum systems the Runge–Lenz vector provides a very elegant means for obtaining the full quantum picture of the system (as in the case of the hydrogen atom).

Over the years this vector, its generalizations and the symmetries associated with it were regarded more as a curiosity than as a valuable tool which is capable of providing innovative results. Their existence was used to illuminate interesting aspects in the systems in which they were found, but they never provided essentially new information. Here, apparently for the first time, the Runge–Lenz vector plays an essential role in providing new results.

Our results, together with the various relativistic Runge–Lenz-like vectors mentioned in section 1, indicate that any discussion of the relativistic CM should take into account the Runge–Lenz vector and the associated symmetry as part of the internal dynamical symmetries of the system. Composite relativistic systems have been studied so far only via their Lorentz–Poincaré symmetry, with the implied internal rotational symmetry. It seems that whatever this symmetry may tell us has been already used up. In order to really proceed and deepen our understanding of these systems larger symmetries which are available must be used, and the generalization of the classical Runge–Lenz symmetry is the natural candidate.

The virtue of the method used here for the computation of the relativistic CM is in the realization that the Runge–Lenz vector naturally appears in the computation of the Lorentz boost. Several directions are open for further investigation:

(i) The Newtonian Runge–Lenz vector for $1/r$ potentials (equation (14)) is of a particularly simple form, neatly separated into a kinetic part $\vec{p} \times \vec{\ell}$ and an interaction part. Generalized

Runge–Lenz vectors exist for all central potentials [7], but this structural simplicity is lost for potentials other than $1/r$ [9, 10]. Further work, done after the completion of the present paper, shows that the above procedure may be applied also to the post-Newtonian extensions of general central potentials [23]. This will be the subject of a future publication.

(ii) Although the general Runge–Lenz symmetry—$SO(4)$ or $SO(3, 1)$ together with $SU(3)$ symmetries—may be shown to exist for general rotationally symmetric systems with arbitrary number of particles [6–8], we know how to construct Runge–Lenz vectors only for two-body systems. This presents the challenge of generalizing the Runge–Lenz symmetry also for systems with more than two particles, and again see if the above procedure works also in this case.

(iii) The ultimate goal is the explicit definition of the CM coordinate for a fully relativistic system and the consequent separation of its internal dynamics from the CM motion. So far, no exact analytic solution is known for closed interacting relativistic systems except for circularly moving charged particles [24, 25]. The Runge–Lenz vector vanishes for circular motion, and in the absence of a full non-circular solution one might try to consider applying the above procedure to almost circular orbits [26] for which the Runge–Lenz vector may be found as first-order correction.

I hope to see some progress in these directions in the near future.

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